

## FOURIER FRAMES FOR THE CANTOR-4 SET

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**ABSTRACT.** The measure supported on the Cantor-4 set constructed by Jorgensen-Pedersen is known to have a Fourier basis, i.e. that it possess a sequence of exponentials which form an orthonormal basis. We construct Fourier frames for this measure via a dilation theory type construction. We expand the Cantor-4 set to a 2 dimensional fractal which admits a representation of a Cuntz algebra. Using the action of this algebra, an orthonormal set is generated on the larger fractal, which is then projected onto the Cantor-4 set to produce a Fourier frame.

Jorgensen and Pedersen [10] demonstrated that there exist singular measures  $\nu$  which are spectral—that is, they possess a sequence of exponential functions which form an orthonormal basis in  $L^2(\nu)$ . The canonical example of such a singular and spectral measure is the uniform measure on the Cantor 4-set defined as follows:

$$C_4 = \{x \in [0, 1] : x = \sum_{k=1}^{\infty} \frac{a_k}{4^k}, a_k \in \{0, 2\}\}.$$

This is analogous to the standard middle third Cantor set where  $4^k$  replaces  $3^k$ . The set  $C_4$  can also be described as the attractor set of the following iterated function system on  $\mathbb{R}$ :

$$\tau_0(x) = \frac{x}{4}, \quad \tau_2(x) = \frac{x+2}{4}.$$

The uniform measure on the set  $C_4$  then is the unique probability measure  $\mu_4$  which is invariant under this iterated function system:

$$\int f(x) d\mu_4(x) = \frac{1}{2} \left( \int f(\tau_0(x)) d\mu_4(x) + \int f(\tau_2(x)) d\mu_4(x) \right)$$

for all  $f \in C(\mathbb{R})$ , see [9] for details. The standard spectrum for  $\mu_4$  is  $\Gamma_4 = \{\sum_{n=0}^N l_n 4^n : l_n \in \{0, 1\}\}$ , though there are many spectra [4, 2].

Remarkably, Jorgensen and Pedersen prove that the uniform measure  $\mu_3$  on the standard middle third Cantor set is not spectral. Indeed, there are no three mutually orthogonal exponentials in  $L^2(\mu_3)$ . Thus, there has been much attention on whether there exists a Fourier frame for  $L^2(\mu_3)$ —the problem is still unresolved, but see [5, 6] for progress in this regard. In this paper, we will construct Fourier frames for  $L^2(\mu_4)$  using a dilation theory type argument. The motivation is whether the construction we demonstrate here for  $\mu_4$  will be applicable to  $\mu_3$ . Fourier frames for  $\mu_4$  were constructed in [6] using a duality type construction.

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A frame for a Hilbert space  $H$  is a sequence  $\{x_n\}_{n \in I} \subset H$  such that there exists constants  $A, B > 0$  such that for all  $v \in H$ ,

$$A\|v\|^2 \leq \sum_{n \in I} |\langle v, x_n \rangle|^2 \leq B\|v\|^2.$$

The largest  $A$  and smallest  $B$  which satisfy these inequalities are called the frame bounds. The frame is called a Parseval frame if both frame bounds are 1. The sequence  $\{x_n\}_{n \in I}$  is a Bessel sequence if there exists a constant  $B$  which satisfies the second inequality, whether or not the first inequality holds;  $B$  is called the Bessel bound. A Fourier frame for  $L^2(\mu_4)$  is a sequence of frequencies  $\{\lambda_n\}_{n \in I} \subset \mathbb{R}$  together with a sequence of “weights”  $\{d_n\}_{n \in I} \subset \mathbb{C}$  such that  $x_n = d_n e^{2\pi i \lambda_n x}$  is a frame. Fourier frames (unweighted) for Lebesgue measure were introduced by Duffin and Schaffer [3], see also Ortega-Cerda and Seip [13].

It was proven in [8] that a frame for a Hilbert space can be dilated to a Riesz basis for a bigger space, that is to say, that any frame is the image under a projection of a Riesz basis. Moreover, a Parseval frame is the image of an orthonormal basis under a projection. This result is now known to be a consequence of the Naimark dilation theory. This will be our recipe for constructing a Fourier frame: constructing a basis in a bigger space and then projecting onto a subspace. We require the following result along these lines [1]:

**Lemma 1.** *Let  $H$  be a Hilbert space,  $V, K$  closed subspaces, and let  $P_V$  be the projection onto  $V$ . If  $\{x_n\}_{n \in I}$  is a frame in  $K$  with frame bounds  $A, B$ , then:*

1.  *$\{P_V x_n\}_{n \in I}$  is a Bessel sequence in  $V$  with Bessel bound no greater than  $B$ ;*
2. *if the projection  $P_V : K \rightarrow V$  is onto, then  $\{P_V x_n\}_{n \in I}$  is a frame in  $V$ ;*
3. *if  $V \subset K$ , then  $\{P_V x_n\}_{n \in I}$  is a frame in  $V$  with frame bounds between  $A$  and  $B$ .*

Note that if  $V \subset K$  and  $\{x_n\}_{n \in I}$  is a Parseval frame for  $K$ , then  $\{P_V x_n\}_{n \in I}$  is a Parseval frame for  $V$ . In the second item above, it is possible that the lower frame bound for  $\{P_V x_n\}$  is smaller than  $A$ , but the upper frame bound is still no greater than  $B$ .

The foundation of our construction is a dilation theory type argument. Our first step, described in Section 1, is to consider the fractal like set  $C_4 \times [0, 1]$ , which we will view in terms of an iterated function system. This IFS will give rise to a representation of the Cuntz algebra  $\mathcal{O}_4$  on  $L^2(\mu_4 \times \lambda)$  since  $\mu_4 \times \lambda$  is the invariant measure under the IFS. Then in Section 2, we will generate via the action of  $\mathcal{O}_4$  an orthonormal set in  $L^2(\mu_4 \times \lambda)$  whose vectors have a particular structure. In Section 3, we consider a subspace  $V$  of  $L^2(\mu_4 \times \lambda)$  which can be naturally identified with  $L^2(\mu_4)$ , and then project the orthonormal set onto  $V$  to, ultimately, obtain a frame. Of paramount importance will be whether the orthonormal set generated by  $\mathcal{O}_4$  spans the subspace  $V$  so that the projection yields a Parseval frame. Section 4 demonstrates concrete constructions in which this occurs, and identifies all possible Fourier frames that can be constructed using this method.

We note here that there may be Fourier frames for  $L^2(\mu_4)$  which cannot be constructed in this manner, but we are unaware of such an example.

## 1. DILATION OF THE CANTOR-4 SET

We wish to construct a Hilbert space  $H$  which contains  $L^2(\mu_4)$  as a subspace in a natural way. We will do this by making the fractal  $C_4$  bigger as follows. We begin with an iterated

function system on  $\mathbb{R}^2$  given by:

$$\Upsilon_0(x, y) = \left(\frac{x}{4}, \frac{y}{2}\right), \quad \Upsilon_1(x, y) = \left(\frac{x+2}{4}, \frac{y}{2}\right), \quad \Upsilon_2(x, y) = \left(\frac{x}{4}, \frac{y+1}{2}\right), \quad \Upsilon_3(x, y) = \left(\frac{x+2}{4}, \frac{y+1}{2}\right).$$

As these are contractions on  $\mathbb{R}^2$ , there exists a compact attractor set, which is readily verified to be  $C_4 \times [0, 1]$ . Likewise, by Hutchinson [9], there exists an invariant probability measure supported on  $C_4 \times [0, 1]$ ; it is readily verified that this invariant measure is  $\mu_4 \times \lambda$ , where  $\lambda$  denotes the Lebesgue measure restricted to  $[0, 1]$ . Thus, for every continuous function  $f : \mathbb{R}^2 \rightarrow \mathbb{C}$ ,

$$(1) \quad \int f(x, y) d(\mu_4 \times \lambda) = \frac{1}{4} \left( \int f\left(\frac{x}{4}, \frac{y}{2}\right) d(\mu_4 \times \lambda) + \int f\left(\frac{x+2}{4}, \frac{y}{2}\right) d(\mu_4 \times \lambda) \right. \\ \left. + \int f\left(\frac{x}{4}, \frac{y+1}{2}\right) d(\mu_4 \times \lambda) + \int f\left(\frac{x+2}{4}, \frac{y+1}{2}\right) d(\mu_4 \times \lambda) \right).$$

The iterated function system  $\Upsilon_j$  has a left inverse on  $C_4 \times [0, 1]$ , given by

$$R : C_4 \times [0, 1] \rightarrow C_4 \times [0, 1] : (x, y) \mapsto (4x, 2y) \pmod{1},$$

so that  $R \circ \Upsilon_j(x, y) = (x, y)$  for  $j = 0, 1, 2, 3$ .

We will use the iterated function system to define an action of the Cuntz algebra  $\mathcal{O}_4$  on  $L^2(\mu_4 \times \lambda)$ . To do so, we choose filters

$$\begin{aligned} m_0(x, y) &= H_0(x, y) \\ m_1(x, y) &= e^{2\pi i x} H_1(x, y) \\ m_2(x, y) &= e^{4\pi i x} H_2(x, y) \\ m_3(x, y) &= e^{6\pi i x} H_3(x, y) \end{aligned}$$

where

$$H_j(x, y) = \sum_{k=0}^3 a_{jk} \chi_{\Upsilon_k(C_4 \times [0, 1])}(x, y)$$

for some choice of scalar coefficients  $a_{jk}$ . In order to obtain a representation of  $\mathcal{O}_4$  on  $L^2(\mu_4 \times \lambda)$ , we require that the above filters satisfy the matrix equation  $\mathcal{M}^*(x, y)\mathcal{M}(x, y) = I$  for  $\mu_4 \times \lambda$  almost every  $(x, y)$ , where

$$\mathcal{M}(x, y) = \begin{pmatrix} m_0(\Upsilon_0(x, y)) & m_0(\Upsilon_1(x, y)) & m_0(\Upsilon_2(x, y)) & m_0(\Upsilon_3(x, y)) \\ m_1(\Upsilon_0(x, y)) & m_1(\Upsilon_1(x, y)) & m_1(\Upsilon_2(x, y)) & m_1(\Upsilon_3(x, y)) \\ m_2(\Upsilon_0(x, y)) & m_2(\Upsilon_1(x, y)) & m_2(\Upsilon_2(x, y)) & m_2(\Upsilon_3(x, y)) \\ m_3(\Upsilon_0(x, y)) & m_3(\Upsilon_1(x, y)) & m_3(\Upsilon_2(x, y)) & m_3(\Upsilon_3(x, y)) \end{pmatrix}$$

For our choice of filters, the matrix  $\mathcal{M}$  becomes

$$\mathcal{M}(x, y) = \begin{pmatrix} a_{00} & a_{01} & a_{02} & a_{03} \\ e^{\pi i x/2} a_{10} & -e^{\pi i x/2} a_{11} & e^{\pi i x/2} a_{12} & -e^{\pi i x/2} a_{13} \\ e^{\pi i x} a_{20} & e^{\pi i x} a_{21} & e^{\pi i x} a_{22} & e^{\pi i x} a_{23} \\ e^{3\pi i x/2} a_{30} & -e^{3\pi i x/2} a_{31} & e^{3\pi i x/2} a_{32} & -e^{3\pi i x/2} a_{33} \end{pmatrix},$$

which is unitary if and only if the matrix

$$H = \begin{pmatrix} a_{00} & a_{01} & a_{02} & a_{03} \\ a_{10} & -a_{11} & a_{12} & -a_{13} \\ a_{20} & a_{21} & a_{22} & a_{23} \\ a_{30} & -a_{31} & a_{32} & -a_{33} \end{pmatrix}$$

is unitary. For the remainder of this section, we assume that  $H$  is unitary.

**Lemma 2.** *The operator  $S_j : L^2(\mu_4 \times \lambda) \rightarrow L^2(\mu_4 \times \lambda)$  given by*

$$[S_j f](x, y) = \sqrt{4} m_j(x, y) f(R(x, y))$$

*is an isometry.*

*Proof.* We calculate:

$$\begin{aligned} \|S_j f\|^2 &= \int |\sqrt{4} m_j(x, y) f(R(x, y))|^2 d(\mu_4 \times \lambda) \\ &= \frac{1}{4} \sum_{k=0}^3 \int 4 |m_j(\Upsilon_k(x, y)) f(R(\Upsilon_k(x, y)))|^2 d(\mu_4 \times \lambda) \\ &= \int \left( \sum_{k=0}^3 |m_j(\Upsilon_k(x, y))|^2 \right) |f(x, y)|^2 d(\mu_4 \times \lambda). \end{aligned}$$

We used Equation (1) in the second line. The sum in the integral is the square of the Euclidean norm of the  $j$ -th row of the matrix  $\mathcal{M}$ , which is unitary. Hence, the sum is 1, so the integral is  $\|f\|^2$ , as required.  $\square$

**Lemma 3.** *The adjoint is given by*

$$[S_j^* f](x, y) = \frac{1}{2} \sum_{k=0}^3 \overline{m_j(\Upsilon_k(x, y))} f(\Upsilon_j(x, y)).$$

*Proof.* Let  $f, g \in L^2(\mu_4 \times \lambda)$ . We calculate

$$\begin{aligned} \langle S_j f, g \rangle &= \int \sqrt{4} m_j(x, y) f(R(x, y)) \overline{g(x, y)} d(\mu_4 \times \lambda) \\ &= \frac{1}{4} \sum_{k=0}^3 \int \sqrt{4} m_j(\Upsilon_k(x, y)) f(R(\Upsilon_k(x, y))) \overline{g(\Upsilon_k(x, y))} d(\mu_4 \times \lambda) \\ &= \int f(x, y) \overline{\left( \frac{1}{2} \sum_{k=0}^3 \overline{m_j(\Upsilon_k(x, y))} g(\Upsilon_k(x, y)) \right)} d(\mu_4 \times \lambda) \end{aligned}$$

where we use Equation (1) and the fact that  $R$  is a left inverse of  $\Upsilon_k$ .  $\square$

**Lemma 4.** *The isometries  $S_j$  satisfy the Cuntz relations:*

$$S_j^* S_k = \delta_{jk} I, \quad \sum_{k=0}^3 S_k S_k^* = I.$$

*Proof.* We consider the orthogonality relation first. Let  $f \in L^2(\mu_4 \times \lambda)$ . We calculate:

$$\begin{aligned} [S_j^* S_k f](x, y) &= \frac{1}{2} \sum_{\ell=0}^3 \overline{m_j(\Upsilon_\ell(x, y))} [S_k f](\Upsilon_\ell(x, y)) \\ &= \frac{1}{2} \sum_{\ell=0}^3 \overline{m_j(\Upsilon_\ell(x, y))} \sqrt{4} m_k(\Upsilon_\ell(x, y)) f(R(\Upsilon_\ell(x, y))) \\ &= \left( \sum_{\ell=0}^3 \overline{m_j(\Upsilon_\ell(x, y))} m_k(\Upsilon_\ell(x, y)) \right) f(x, y). \end{aligned}$$

Note that the sum is the scalar product of the  $k$ -th row with the  $j$ -th row of the matrix  $\mathcal{M}$ , which is unitary. Hence, the sum is  $\delta_{jk}$  as required.

Now for the identity relation, let  $f, g \in L^2(\mu_4 \times \lambda)$ . We calculate:

$$\begin{aligned} \left\langle \sum_{k=0}^3 S_k S_k^* f, g \right\rangle &= \sum_{k=0}^3 \langle S_k^* f, S_k^* g \rangle \\ &= \sum_{k=0}^3 \int \left( \frac{1}{2} \sum_{\ell=0}^3 \overline{m_k(\Upsilon_\ell(x, y))} f(\Upsilon_\ell(x, y)) \right) \left( \frac{1}{2} \sum_{n=0}^3 \overline{m_k(\Upsilon_n(x, y))} g(\Upsilon_n(x, y)) \right) d(\mu_4 \times \lambda) \\ &= \sum_{\ell=0}^3 \sum_{n=0}^3 \frac{1}{4} \int \left( \sum_{k=0}^3 \overline{m_k(\Upsilon_\ell(x, y))} m_k(\Upsilon_n(x, y)) \right) f(\Upsilon_\ell(x, y)) \overline{g(\Upsilon_n(x, y))} d(\mu_4 \times \lambda) \\ &= \frac{1}{4} \sum_{n=0}^3 \int f(\Upsilon_n(x, y)) \overline{g(\Upsilon_n(x, y))} d(\mu_4 \times \lambda) \\ &= \int f(x, y) \overline{g(x, y)} d(\mu_4 \times \lambda) \\ &= \langle f, g \rangle. \end{aligned}$$

Note that the sum over  $k$  in the third line is the scalar product of the  $\ell$ -th column with the  $n$ -th column of  $\mathcal{M}$ , so the sum collapses to  $\delta_{\ell n}$ . The sum on  $n$  in the fourth line collapses by Equation (1).  $\square$

## 2. ORTHONORMAL SETS IN $L^2(\mu_4 \times \lambda)$

Since the isometries  $S_j$  satisfy the Cuntz relations, we can use them to generate orthonormal sets in the space  $L^2(\mu_4 \times \lambda)$ . We do so by having the isometries act on a generating vector. We consider words in the alphabet  $\{0, 1, 2, 3\}$ ; let  $W_4$  denote the set of all such words. For a word  $\omega = j_K j_{K-1} \dots j_1$ , we denote by  $|\omega| = K$  the length of the word, and define

$$S_\omega f = S_{j_K} S_{j_{K-1}} \dots S_{j_1} f.$$

**Definition 1.** Let

$$X_4 = \{\omega \in W_4 : |\omega| = 1\} \cup \{\omega \in W_4 : |\omega| \geq 2, j_1 \neq 0\}.$$

For convenience, we allow the empty word  $\omega_\emptyset$  with length 0, and define  $S_{\omega_\emptyset} = I$ , the identity.

**Lemma 5.** *Suppose  $f \in L^2(\mu_4 \times \lambda)$  with  $\|f\| = 1$ , and that  $S_0 f = f$ . Then,*

$$\{S_\omega f : \omega \in X_4\}$$

*is an orthonormal set.*

*Proof.* Suppose  $\omega, \omega' \in X_4$  with  $\omega \neq \omega'$ . First consider  $|\omega| = |\omega'|$ , with  $\omega = j_K \dots j_1$  and  $\omega' = i_K \dots i_1$ . Suppose that  $\ell$  is the largest index such that  $j_\ell \neq i_\ell$ . Then we have

$$\langle S_\omega f, S_{\omega'} f \rangle = \langle S_{j_\ell} \dots S_{j_1} f, S_{i_\ell} \dots S_{i_1} f \rangle = \langle S_{i_\ell}^* S_{j_\ell} \dots S_{j_1} f, S_{i_{\ell-1}} \dots S_{i_1} f \rangle = 0$$

by the orthogonality condition of the Cuntz relations.

Now, if  $K = |\omega| > |\omega'| = M$ , with  $\omega' = i_M \dots i_1$ , we define the word  $\rho = i_M \dots i_1 0 \dots 0$  so that  $|\rho| = K$ . Note that  $\rho \notin X_4$  so  $\omega \neq \rho$ . Note further that  $S_{\omega'} f = S_\rho f$ . Thus, by a similar argument to that above, we have

$$\langle S_\omega f, S_{\omega'} f \rangle = 0.$$

□

*Remark 1.* The set  $\{S_\omega f : \omega \in X_4\}$  need not be complete. We will provide an example of this in Example 1 in Section 4.

Our goal is to project the set  $\{S_\omega f : \omega \in X_4\}$  onto some subspace  $V$  of  $L^2(\mu_4 \times \lambda)$  to obtain a frame. To that end, we need to know when the projection  $\{P_V S_\omega f : \omega \in X_4\}$  is a frame, which by Lemma 1 requires the projection  $P_V : K \rightarrow V$  to be onto, where  $K$  is the subspace spanned by  $\{S_\omega f : \omega \in X_4\}$ . The tool we will use is the following result, which is a minor adaptation of a result from [7]; we will not use this result directly, but will use all of the critical components.

**Theorem 1.** *Let  $\mathcal{H}$  be a Hilbert space,  $\mathcal{K} \subset \mathcal{H}$  a closed subspace, and  $(S_i)_{i=0}^{N-1}$  be a representation of the Cuntz algebra  $\mathcal{O}_N$ . Let  $\mathcal{E}$  be an orthonormal set in  $\mathcal{H}$  and  $f : X \rightarrow \mathcal{K}$  a norm continuous function on a topological space  $X$  with the following properties:*

- i)  $\mathcal{E} = \cup_{i=0}^{N-1} S_i \mathcal{E}$  where the union is disjoint.
  - ii)  $\overline{\text{span}}\{f(t) : t \in X\} = \mathcal{K}$  and  $\|f(t)\| = 1$ , for all  $t \in X$ .
  - iii) There exist functions  $\mathbf{m}_i : X \rightarrow \mathbb{C}$ ,  $g_i : X \rightarrow X$ ,  $i = 0, \dots, N-1$  such that
- $$(2) \quad S_i^* f(t) = \mathbf{m}_i(t) f(g_i(t)), \quad t \in X.$$
- iv) There exist  $c_0 \in X$  such that  $f(c_0) \in \overline{\text{span}} \mathcal{E}$ .
  - v) The only function  $h \in \mathcal{C}(X)$  with  $h \geq 0$ ,  $h(c) = 1$ ,  $\forall c \in \{x \in X : f(x) \in \overline{\text{span}} \mathcal{E}\}$ , and

$$(3) \quad h(t) = \sum_{i=0}^{N-1} |\mathbf{m}_i(t)|^2 h(g_i(t)), \quad t \in X$$

*are the constant functions.*

*Then  $\mathcal{K} \subset \overline{\text{span}} \mathcal{E}$ .*

### 3. THE PROJECTION

Recall the definition of the filters  $m_j(x, y) = e^{2\pi i j x} H_j(x, y)$  from Section 1. We choose the filter coefficients  $a_{jk}$  so that the matrix  $H$  is unitary. We place the additional constraint that

$$a_{00} = a_{01} = a_{02} = a_{03} = \frac{1}{2},$$

so that  $S_0\mathbb{1} = \mathbb{1}$ , where  $\mathbb{1}$  the function in  $L^2(\mu_4 \times \lambda)$  which is identically 1. As  $S_0\mathbb{1} = \mathbb{1}$ , by Lemma 5, the set  $\{S_\omega\mathbb{1} : \omega \in X_4\}$  is orthonormal. Moreover, we place the additional constraint that for every  $j$ ,  $a_{j0} + a_{j2} = a_{j1} + a_{j3}$ , which will be required for our calculation of the projection.

**Definition 2.** We define the subspace  $V = \{f \in L^2(\mu_4 \times \lambda) : f(x, y) = g(x)\chi_{[0,1]}(y), g \in L^2(\mu_4)\}$ . Note that the subspace  $V$  can be identified with  $L^2(\mu_4)$  via the isometric isomorphism  $g \mapsto g(x)\chi_{[0,1]}(y)$ . We will suppress the  $y$  variable in the future.

**Definition 3.** We define a function  $c : X_4 \rightarrow \mathbb{N}_0$  as follows: for a word  $\omega = j_K j_{K-1} \dots j_1$ ,

$$c(\omega) = \sum_{k=1}^K j_k 4^{K-k}.$$

Here  $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ . It is readily verified that  $c$  is a bijection.

**Lemma 6.** For a word  $\omega = j_K j_{K-1} \dots j_1$ ,

$$S_\omega\mathbb{1} = e^{2\pi i c(\omega)x} \left( \prod_{k=1}^K 2H_{j_k}(R^{K-k}(x, y)) \right).$$

*Proof.* We proceed by induction on the length of the word  $\omega$ . The equality is readily verified for  $|\omega| = 1$ . Let  $\omega_0 = j_{K-1} j_{K-2} \dots j_1$ . We have

$$\begin{aligned} S_\omega\mathbb{1} &= S_{j_K} S_{\omega_0}\mathbb{1} \\ &= S_{j_K} \left[ e^{2\pi i c(\omega_0)x} \left( \prod_{k=1}^{K-1} 2H_{j_k}(R^{K-1-k}(x, y)) \right) \right] \\ &= 2e^{2\pi i \lambda_{j_K} x} H_{j_K}(x, y) e^{2\pi i c(\omega_0) \cdot 4x} \left( \prod_{k=1}^{K-1} H_{j_k}(R^{K-k}(x, y)) \right) \\ &= 2e^{2\pi i (\lambda_{j_K} + 4c(\omega_0))x} H_{j_K}(R^{K-K}(x, y)) \left( \prod_{k=1}^{K-1} 2H_{j_k}(R^{K-k}(x, y)) \right) \\ &= e^{2\pi i c(\omega)x} \left( \prod_{k=1}^K 2H_{j_k}(R^{K-k}(x, y)) \right). \end{aligned}$$

The last line above is justified by the following calculation:

$$\begin{aligned} \lambda_{j_K} + 4c(\omega_0) &= \lambda_{j_K} + 4 \left( \sum_{k=1}^{K-1} \lambda_{j_k} 4^{K-1-k} \right) \\ &= \lambda_{j_K} 4^{K-K} + \sum_{k=1}^{K-1} \lambda_{j_k} 4^{K-k} \\ &= \sum_{k=1}^K \lambda_{j_k} 4^{K-k} \\ &= c(\omega). \end{aligned}$$

□

We wish to project the vectors  $S_\omega \mathbb{1}$  onto the subspace  $V$ . The following lemma calculates that projection, where  $P_V$  denotes the projection onto the subspace  $V$ .

**Lemma 7.** *If  $f(x, y) = g(x)h(x, y)$  with  $g \in L^2(\mu_4)$  and  $h \in L^\infty(\mu_4 \times \lambda)$ , then*

$$[P_V f](x, y) = g(x)G(x)$$

where  $G(x) = \int_{[0,1]} h(x, y) d\lambda(y)$ .

*Proof.* We verify that for every  $F(x) \in L^2(\mu_4)$ ,  $f(x, y) - g(x)G(x)$  is orthogonal to  $F(x)$ . We calculate utilizing Fubini's theorem:

$$\begin{aligned} \langle f - gG, F \rangle &= \int \int g(x)h(x, y) \overline{F(x)} d(\mu_4 \times \lambda) - \int \int g(x)G(x) \overline{F(x)} d(\mu_4 \times \lambda) \\ &= \int_{C_4} g(x) \overline{F(x)} \left( \int_{[0,1]} h(x, y) - G(x) d\lambda(y) \right) d\mu_4(x) \\ &= \int_{C_4} g(x) \overline{F(x)} (G(x) - G(x)) d\mu_4(x) \\ &= 0. \end{aligned}$$

□

For the purposes of the following lemma,  $\alpha x$  and  $\beta y$  are understood to be modulo 1.

**Lemma 8.** *For any word  $\omega = j_K j_{K-1} \dots j_1$ ,*

$$\int \prod_{k=1}^K 2H_{j_k}(R^{k-1}(x, y)) d\lambda(y) = \prod_{k=1}^K 2 \int H_{j_k}(4^{k-1}x, y) d\lambda(y).$$

*Proof.* Let  $F_m(x, y) = \prod_{k=m}^K 2H_{j_k}(4^{k-1}x, 2^{k-m}y)$ . Note that

$$F_m(x, \frac{y}{2}) = 2H_{j_m}(4^{m-1}x, \frac{y}{2}) \left( \prod_{k=m+1}^K 2H_{j_k}(4^{k-1}x, 2^{k-(m+1)}y) \right) = 2H_{j_m}(4^{m-1}x, \frac{y}{2}) F_{m+1}(x, y).$$

Likewise for  $F_m(x, \frac{y+1}{2})$ .

Since  $\lambda$  is the invariant measure for the iterated function system  $y \mapsto \frac{y}{2}$ ,  $y \mapsto \frac{y+1}{2}$ , we calculate:

$$\begin{aligned} \int_0^1 F_m(x, y) d\lambda(y) &= \frac{1}{2} \left[ \int_0^1 F_m(x, \frac{y}{2}) d\lambda(y) + \int_0^1 F_m(x, \frac{y+1}{2}) d\lambda(y) \right] \\ &= \frac{1}{2} \left[ \int_0^1 2H_{j_m}(4^{m-1}x, \frac{y}{2}) F_{m+1}(x, y) + 2H_{j_m}(4^{m-1}x, \frac{y+1}{2}) F_{m+1}(x, y) d\lambda(y) \right] \\ &= \frac{1}{2} \left[ \int_0^1 2a_{j_m, q} F_{m+1}(x, y) + 2a_{j_m, q+2} F_{m+1}(x, y) d\lambda(y) \right] \\ &= \frac{1}{2} [2a_{j_m, q} + 2a_{j_m, q+2}] \cdot \left[ \int_0^1 F_{m+1}(x, y) d\lambda(y) \right] \\ &= \left[ \int_0^1 2H_{j_m}(4^{m-1}x, y) d\lambda(y) \right] \cdot \left[ \int_0^1 F_{m+1}(x, y) d\lambda(y) \right] \end{aligned}$$

where  $q = 0$  if  $0 \leq 4^{m-1}x < \frac{1}{2}$ , and  $q = 1$  if  $\frac{1}{2} \leq 4^{m-1}x < 1$ .



The result now follows by a standard induction argument. □

**Proposition 1.** *Suppose the filters  $m_j(x, y)$  are chosen so that*

- i) *the matrix  $H$  is unitary,*
- ii)  *$a_{00} = a_{01} = a_{02} = a_{03} = \frac{1}{2}$ , and*
- iii) *for  $j = 0, 1, 2, 3$ ,  $a_{j0} + a_{j2} = a_{j1} + a_{j3}$ .*

*Then for any word  $\omega = j_K \dots j_1$ ,*

$$P_V S_\omega \mathbf{1} = d_\omega e^{2\pi i c(\omega)x},$$

*where*

$$(4) \quad d_\omega = \prod_{k=1}^K (a_{j_k 0} + a_{j_k 2}).$$

*Proof.* We apply the previous three Lemmas to obtain

$$\begin{aligned} [P_V S_\omega \mathbf{1}](x, y) &= e^{2\pi i c(\omega)x} \int \prod_{k=1}^K 2H_{j_k}(4^{k-1}x, y) d\lambda(y) \\ &= e^{2\pi i c(\omega)x} \prod_{k=1}^K 2 \int H_{j_k}(4^{k-1}x, y) d\lambda(y) \end{aligned}$$

By assumption iii), the integral  $\int H_{j_k}(4^{k-1}x, y) d\lambda(y)$  is independent of  $x$ , and the value of the integral is  $\frac{a_{j0}}{2} + \frac{a_{j2}}{2}$ . Equation 4 now follows. □

#### 4. CONCRETE CONSTRUCTIONS

We now turn to concrete constructions of Fourier frames for  $\mu_4$ . The hypotheses of Lemma 5 and Proposition 1 require  $H$  to be unitary and requires the matrix

$$A = \begin{pmatrix} a_{00} & a_{01} & a_{02} & a_{03} \\ a_{10} & a_{11} & a_{12} & a_{13} \\ a_{20} & a_{21} & a_{22} & a_{23} \\ a_{30} & a_{31} & a_{32} & a_{33} \end{pmatrix}$$

to have the first row be identically  $\frac{1}{2}$  and to have the vector  $(1 \ -1 \ 1 \ -1)^T$  in the kernel.

We can use Hadamard matrices to construct examples of such a matrix  $A$ . Every  $4 \times 4$  Hadamard matrix is a permutation of the following matrix:

$$U_\rho = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & \rho & -\rho \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -\rho & \rho \end{pmatrix}$$

where  $\rho$  is any complex number of modulus 1.

If we set  $H = U_\rho$ , we obtain

$$(5) \quad A = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & \rho & \rho \\ 1 & 1 & -1 & -1 \\ 1 & 1 & -\rho & -\rho \end{pmatrix}$$

which has the requisite properties to apply Lemma 5 and Proposition 1.

We define for  $k = 1, 2, 3$ ,  $l_k : \mathbb{N}_0 \rightarrow \mathbb{N}_0$  by  $l_k(n)$  is the number of digits equal to  $k$  in the base 4 expansion of  $n$ . Note that  $l_k(0) = 0$ , and we follow the convention that  $0^0 = 1$ .

**Theorem 2.** *For the choice  $A$  as in Equation (5) with  $\rho \neq -1$ , the sequence*

$$(6) \quad \left\{ \left( \frac{1+\rho}{2} \right)^{l_1(n)} 0^{l_2(n)} \left( \frac{1-\rho}{2} \right)^{l_3(n)} e^{2\pi i n x} : n \in \mathbb{N}_0 \right\}$$

*is a Parseval frame in  $L^2(\mu_4)$ .*

*Proof.* By Lemma 5, we have that  $\{S_\omega \mathbb{1} : \omega \in X_4\}$  is an orthonormal set. For a word  $\omega = j_K j_{K-1} \dots j_1$ , Proposition 1 yields that

$$P_V S_\omega \mathbb{1} = e^{2\pi i c(\omega)x} \prod_{k=1}^K (a_{j_k 0} + a_{j_k 2}).$$

Then, setting  $n = c(\omega)$ , we obtain

$$P_V S_\omega \mathbb{1} = e^{2\pi i n x} (a_{00} + a_{02})^{K-l_1(n)-l_2(n)-l_3(n)} \prod_{j=1}^3 (a_{j0} + a_{j2})^{l_j(n)}.$$

Since

$$a_{00} + a_{02} = 1, \quad a_{10} + a_{12} = \frac{1+\rho}{2}, \quad a_{20} + a_{22} = 0, \quad a_{30} + a_{32} = \frac{1-\rho}{2},$$

it follows that

$$P_V S_\omega \mathbb{1} = \left( \frac{1+\rho}{2} \right)^{l_1(n)} 0^{l_2(n)} \left( \frac{1-\rho}{2} \right)^{l_3(n)} e^{2\pi i n x}.$$

Since  $c$  is a bijection, the set  $\{P_V S_\omega \mathbb{1} : \omega \in X_4\}$  coincides with the set in (6).

In order to establish that the set (6) is a Parseval frame, we wish to apply Lemma 1, which requires that the subspace  $V$  is contained in the closed span of  $\{S_\omega \mathbb{1} : \omega \in X_4\}$ . Denote the closed span by  $\mathcal{K}$ . We will proceed in a manner similar to Theorem 1. Define the function  $f : \mathbb{R} \rightarrow V$  by  $f(t) = e_t$  where  $e_t(x, y) = e^{2\pi i x t}$ . Note that  $f(0) = \mathbb{1} \in \mathcal{K}$ . Likewise, define a function  $h_X : \mathbb{R} \rightarrow \mathbb{R}$  by

$$h_X(t) = \sum_{\omega \in X_4} |\langle f(t), S_\omega \mathbb{1} \rangle|^2 = \|P_{\mathcal{K}} f(t)\|^2.$$

**Claim 1.** We have  $h_X \equiv 1$ .

Assuming for the moment that the claim holds, we deduce that  $f(t) \in \mathcal{K}$  for every  $t \in \mathbb{R}$ . Since  $\{f(\gamma) : \gamma \in \Gamma_4\}$  is an orthonormal basis for  $V$ , it follows that the closed span of  $\{f(t) : t \in \mathbb{R}\}$  is all of  $V$ . We conclude that  $V \subset \mathcal{K}$ , and so Lemma 1 implies that  $\{P_V S_\omega \mathbb{1} : \omega \in X_4\}$  is a Parseval frame for  $V$ , from which the Theorem follows.

Thus, we turn to the proof of Claim 1. First, we require  $\{S_\omega \mathbb{1} : \omega \in X_4\} = \cup_{j=0}^3 \{S_j S_\omega \mathbb{1} : \omega \in X_4\}$ , where the union is disjoint. Clearly, the RHS is a subset of the LHS, and the union is disjoint. Consider an element of the LHS:  $S_\omega \mathbb{1}$ . If  $|\omega| \geq 2$ , we write  $S_\omega \mathbb{1} = S_j S_{\omega_0} \mathbb{1}$  for some  $j$  and some  $\omega_0 \in X_4$ , whence  $S_\omega \mathbb{1}$  is in the RHS. If  $|\omega| = 1$ , then we write  $S_\omega \mathbb{1} = S_j \mathbb{1} = S_j S_0 \mathbb{1}$ , which is again an element of the RHS. Equality now follows.

As a consequence,

$$\begin{aligned}
 h_X(t) &= \sum_{\omega \in X_4} |\langle f(t), S_\omega \mathbf{1} \rangle|^2 \\
 &= \sum_{j=0}^3 \sum_{\omega \in X_4} |\langle f(t), S_j S_\omega \mathbf{1} \rangle|^2 \\
 &= \sum_{j=0}^3 \sum_{\omega \in X_4} |\langle S_j^* f(t), S_\omega \mathbf{1} \rangle|^2.
 \end{aligned}$$

We calculate:

$$\begin{aligned}
 [S_j^* f(t)](x, y) &= \frac{1}{2} \sum_{k=0}^3 \overline{m_j(\Upsilon_k(x, y))} e_t(\Upsilon_k(x, y)) \\
 &= \frac{1}{2} \left[ \overline{a_{j0}} e^{-2\pi i j x/4} e_t\left(\frac{x}{4}, \frac{y}{2}\right) + e^{-\pi i j} \overline{a_{j1}} e^{-2\pi i j x/4} e_t\left(\frac{x+2}{4}, \frac{y}{2}\right) \right. \\
 &\quad \left. + \overline{a_{j2}} e^{-2\pi i j x/4} e_t\left(\frac{x}{4}, \frac{y+1}{2}\right) + e^{-\pi i j} \overline{a_{j3}} e^{-2\pi i j x/4} e_t\left(\frac{x+2}{4}, \frac{y+1}{2}\right) \right] \\
 &= \frac{1}{2} \left[ \overline{a_{j0}} e^{-2\pi i j x/4} e_t\left(\frac{x}{4}, \frac{y}{2}\right) + e^{-\pi i j} \overline{a_{j1}} e^{-2\pi i j x/4} e^{\pi i t} e_t\left(\frac{x}{4}, \frac{y}{2}\right) \right. \\
 &\quad \left. + \overline{a_{j2}} e^{-2\pi i j x/4} e_t\left(\frac{x}{4}, \frac{y}{2}\right) + e^{-\pi i j} \overline{a_{j3}} e^{-2\pi i j x/4} e^{\pi i t} e_t\left(\frac{x}{4}, \frac{y}{2}\right) \right] \\
 &= \frac{1}{2} \left[ \overline{a_{j0}} + e^{-\pi i j} \overline{a_{j1}} e^{\pi i t} + \overline{a_{j2}} + e^{-\pi i j} \overline{a_{j3}} e^{\pi i t} \right] e^{-2\pi i j x/4} e_t\left(\frac{x}{4}, \frac{y}{2}\right) \\
 &= \frac{1}{2} \left[ \overline{a_{j0}} + e^{-\pi i j} \overline{a_{j1}} e^{\pi i t} + \overline{a_{j2}} + e^{-\pi i j} \overline{a_{j3}} e^{\pi i t} \right] e^{2\pi i (t \frac{x}{4} - j \frac{x}{4})} \\
 &= \frac{1}{2} \left[ \overline{a_{j0}} + e^{-\pi i j} \overline{a_{j1}} e^{\pi i t} + \overline{a_{j2}} + e^{-\pi i j} \overline{a_{j3}} e^{\pi i t} \right] e^{2\pi i (\frac{t-j}{4} x)} \\
 &= \frac{1}{2} \left[ \overline{a_{j0}} + e^{-\pi i j} \overline{a_{j1}} e^{\pi i t} + \overline{a_{j2}} + e^{-\pi i j} \overline{a_{j3}} e^{\pi i t} \right] e_{\frac{t-j}{4}}(x, y).
 \end{aligned}$$

Thus, we define

$$\mathbf{m}_j(t) = \frac{1}{2} (\overline{a_{j0}} + \overline{a_{j2}}) + \frac{e^{-\pi i j}}{2} (\overline{a_{j1}} + \overline{a_{j3}}) e^{\pi i t},$$

and

$$g_j(t) = \frac{t-j}{4}.$$

As a consequence, we obtain

$$\begin{aligned}
 h_X(t) &= \sum_{j=0}^3 \sum_{\omega \in X_4} |\langle S_j^* f(t), S_\omega \mathbf{1} \rangle|^2 \\
 &= \sum_{j=0}^3 \sum_{\omega \in X_4} |\langle \mathbf{m}_j(t) f(g_j(t)), S_\omega \mathbf{1} \rangle|^2 \\
 (7) \quad &= \sum_{j=0}^3 |\mathbf{m}_j(t)|^2 h_X(g_j(t)).
 \end{aligned}$$

Because of our choice of coefficients in the matrix  $A$ , which has the vector  $(1 \ -1 \ 1 \ -1)^T$  in the kernel, we have for every  $j$ :  $a_{j0} + a_{j2} = a_{j1} + a_{j3}$ . Thus, if we let  $b_j = \overline{a_{j0}} + \overline{a_{j2}}$ , the functions  $\mathbf{m}_j$  simplify to

$$\mathbf{m}_j(t) = b_j e^{\pi i \frac{t}{2}} \cos(\pi \frac{t}{2})$$

for  $j = 0, 2$ , and

$$\mathbf{m}_j(t) = -i b_j e^{\pi i \frac{t}{2}} \sin(\pi \frac{t}{2})$$

for  $j = 1, 3$ . Substituting these into Equation (7),

$$\begin{aligned}
 (8) \quad h_X(t) &= \cos^2\left(\frac{\pi t}{2}\right) h_X\left(\frac{t}{4}\right) + \sin^2\left(\frac{\pi t}{2}\right) \frac{|1 + \overline{\rho}|^2}{2} h_X\left(\frac{t-1}{4}\right) + \sin^2\left(\frac{\pi t}{2}\right) \frac{|1 - \overline{\rho}|^2}{2} h_X\left(\frac{t-3}{4}\right).
 \end{aligned}$$

**Claim 2.** The function  $h_X$  can be extended to an entire function.

Assume for the moment that Claim 2 holds, we finish the proof of Claim 1. If  $h_X(t) = 1$  for  $t \in [-1, 0]$ , then  $h_X(z) = 1$  for all  $z \in \mathbb{C}$ , and Claim 1 holds.

Now, assume to the contrary that  $h_X(t)$  is not identically 1 on  $[-1, 0]$ . Since  $0 \leq h_X(t) \leq 1$  for  $t$  real, then  $\beta = \min\{h_X(t) : t \in [-1, 0]\} < 1$ . Because constant functions satisfy (8),  $h_1 := h_X - \beta$  also satisfies Equation (8). There exists  $t_0$  such that  $h_1(t_0) = 0$  and  $t_0 \neq 0$  as  $h_X(0) = 1$ . Since  $h_1 \geq 0$  each of the terms in (8) must vanish :

$$(9) \quad \cos^2\left(\frac{\pi t_0}{2}\right) h_1\left(\frac{t_0}{4}\right) = 0$$

$$(10) \quad \sin^2\left(\frac{\pi t_0}{2}\right) \frac{|1 + \overline{\rho}|^2}{2} h_1\left(\frac{t_0 - 1}{4}\right) = 0$$

$$(11) \quad \sin^2\left(\frac{\pi t_0}{2}\right) \frac{|1 - \overline{\rho}|^2}{2} h_1\left(\frac{t_0 - 3}{4}\right) = 0$$

Our hypothesis is that  $\rho \neq -1$ , so in Equation (10), the coefficient  $\frac{|1 + \overline{\rho}|}{2} \neq 0$ .

Case 1: If  $t_0 \neq -1$  then Equation (9) implies  $h_1(t_0/4) = 0 = h_1(g_0(t_0))$ . Let  $t_1 := g_0(t_0) \in (-1, 0)$ ; iterating the previous argument implies that  $h_1(g_0(t_1)) = 0$ . Thus, we obtain an infinite sequence of zeroes of  $h_1$ .

Case 2: If  $t_0 = -1$ , then the previous argument does not hold. However, we can construct another zero of  $h_1$ ,  $t'_0 \in (-1, 0)$  to which the previous argument will hold. Indeed, if  $t_0 = -1$ ,

Equation (10) implies  $h_1((t_0 - 1)/4) = h_1(-1/2) = 0$ . Let  $t'_0 = -1/2$  and continue as in Case 1.

In either case,  $h_1$  vanishes on a (countable) set with an accumulation point, and since  $h_1$  is analytic it follows that  $h_1 \equiv 0$ , a contradiction, and Claim 1 holds.

Now, to prove Claim 2, we follow the proof of Lemma 4.2 of [10]. For a fixed  $\omega \in X_4$ , define  $f_\omega : \mathbb{C} \rightarrow \mathbb{C}$  by

$$f_\omega(z) = \langle e_z, S_\omega \mathbf{1} \rangle = \int e^{2\pi i z x} \overline{[S_\omega \mathbf{1}](x, y)} d(\mu_4 \times \lambda).$$

Since the distribution  $\overline{[S_\omega \mathbf{1}](x, y)} d(\mu_4 \times \lambda)$  is compactly supported, a standard convergence argument demonstrates that  $f_\omega$  is entire. Likewise,  $f_\omega^*(z) = \overline{f_\omega(\bar{z})}$  is entire, and for  $t$  real,

$$f_\omega(t)f_\omega^*(t) = (\langle e_t, S_\omega \mathbf{1} \rangle) \left( \overline{\langle e_t, S_\omega \mathbf{1} \rangle} \right) = |\langle e_t, S_\omega \mathbf{1} \rangle|^2.$$

Thus,

$$h_X(t) = \sum_{\omega \in X_4} f_\omega(t)f_\omega^*(t).$$

For  $n \in \mathbb{N}$ , let  $h_n(z) = \sum_{|\omega| \leq n} f_\omega(z)f_\omega^*(z)$ , which is entire. By Hölder's inequality,

$$\begin{aligned} \sum_{\omega \in X_4} |f_\omega(z)f_\omega^*(z)| &\leq \left( \sum_{\omega \in X_4} |\langle e_z, S_\omega \mathbf{1} \rangle|^2 \right)^{1/2} \left( \sum_{\omega \in X_4} |\langle e_{\bar{z}}, S_\omega \mathbf{1} \rangle|^2 \right)^{1/2} \\ &\leq \|e_z\| \|e_{\bar{z}}\| \\ &\leq e^{K \operatorname{Im}(z)} \end{aligned}$$

for some constant  $K$ . Thus, the sequence  $h_n(z)$  converges pointwise to a function  $h(z)$ , and are uniformly bounded on strips  $\operatorname{Im}(z) \leq C$ . By the theorems of Montel and Vitali, the limit function  $h$  is entire, which coincides with  $h_X$  for real  $t$ , and Claim 2 is proved.  $\square$

**Example 1.** As mentioned in Section 2, in general,  $\{S_\omega \mathbf{1}\}$  need not be complete, and the exceptional point  $\rho = -1$  in Theorem 2 provides the example. In the case  $\rho = -1$ , the set (6) becomes

$$\{d_n e^{2\pi i n x} : n \in \mathbb{N}_0\}$$

where the coefficients  $d_n = 1$  if  $n \in \Gamma_3$  and 0 otherwise. Here,

$$\Gamma_3 = \left\{ \sum_{n=0}^N l_n 4^n : l_n \in \{0, 3\} \right\}$$

and it is known [4] that the sequence  $\{e^{2\pi i n x} : n \in \Gamma_3\}$  is incomplete in  $L^2(\mu_4)$ . Thus,  $\{P_V S_\omega \mathbf{1}\}$  is incomplete in  $V$ , so  $\{S_\omega \mathbf{1}\}$  is incomplete in  $L^2(\mu_4 \times \lambda)$ .

We can generalize the construction of Theorem 2 as follows. We want to choose a matrix

$$A = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ h_{10} & h_{11} & h_{12} & h_{13} \\ h_{20} & h_{21} & h_{22} & h_{23} \\ h_{30} & h_{31} & h_{32} & h_{33} \end{pmatrix}$$

such that  $(1 \ -1 \ 1 \ -1)^T$  is in the kernel of  $H$  and the matrix

$$H = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ h_{10} & -h_{11} & h_{12} & -h_{13} \\ h_{20} & h_{21} & h_{22} & h_{23} \\ h_{30} & -h_{31} & h_{32} & -h_{33} \end{pmatrix}$$

is unitary. We obtain a system of nonlinear equations in the 12 unknowns. To parametrize all solutions, we consider the following row vectors:

$$(12) \quad \vec{v}_0 = \frac{1}{2} (1 \ 1 \ 1 \ 1) \quad \vec{w}_0 = \frac{1}{2} (1 \ -1 \ 1 \ -1)$$

$$(13) \quad \vec{v}_1 = \frac{1}{2} (1 \ -1 \ -1 \ 1) \quad \vec{w}_1 = \frac{1}{2} (1 \ 1 \ -1 \ -1)$$

$$(14) \quad \vec{v}_2 = \frac{1}{2} (1 \ 1 \ -1 \ -1) \quad \vec{w}_2 = \frac{1}{2} (1 \ -1 \ -1 \ 1)$$

If we construct the matrix  $A$  so that the rows are linear combinations of  $\{\vec{v}_0, \vec{v}_1, \vec{v}_2\}$ , then  $A$  will satisfy the desired condition on the kernel. Note that if the  $j$ -th row of  $A$  is  $\alpha_{j0}\vec{v}_0 + \alpha_{j1}\vec{v}_1 + \alpha_{j2}\vec{v}_2$  for  $j = 1, 3$ , then the  $j$ -th row of  $H$  is  $\alpha_{j0}\vec{w}_0 + \alpha_{j1}\vec{w}_1 + \alpha_{j2}\vec{w}_2$ , whereas if  $j = 0, 2$ , then the  $j$ -th row of  $H$  is equal to the  $j$ -th row of  $A$ .

Thus, we want to choose coefficients  $\alpha_{jk}$ ,  $j = 0, 1, 2, 3$ ,  $k = 1, 2, 3$  so that the matrix

$$(15) \quad H = \begin{pmatrix} \alpha_{00}\vec{v}_0 + \alpha_{01}\vec{v}_1 + \alpha_{02}\vec{v}_2 \\ \alpha_{10}\vec{w}_0 + \alpha_{11}\vec{w}_1 + \alpha_{12}\vec{w}_2 \\ \alpha_{20}\vec{v}_0 + \alpha_{21}\vec{v}_1 + \alpha_{22}\vec{v}_2 \\ \alpha_{30}\vec{w}_0 + \alpha_{31}\vec{w}_1 + \alpha_{32}\vec{w}_2 \end{pmatrix}$$

is unitary. To satisfy the requirement on the first row, we choose  $\alpha_{00} = 1$  and  $\alpha_{01} = \alpha_{02} = 0$ . Calculating the inner products of the rows of  $H$ , we obtain the following necessary and sufficient conditions:

$$(16) \quad |\alpha_{j0}|^2 + |\alpha_{j1}|^2 + |\alpha_{j2}|^2 = 1$$

$$(17) \quad \alpha_{00}\overline{\alpha_{20}} = 0$$

$$(18) \quad \alpha_{11}\overline{\alpha_{22}} + \alpha_{12}\overline{\alpha_{21}} = 0$$

$$(19) \quad \alpha_{10}\overline{\alpha_{30}} + \alpha_{11}\overline{\alpha_{31}} + \alpha_{12}\overline{\alpha_{32}} = 0$$

$$(20) \quad \alpha_{21}\overline{\alpha_{32}} + \alpha_{22}\overline{\alpha_{31}} = 0$$

**Proposition 2.** *Fix  $\alpha_{00} = 1$ . There exists a solution to the Equations (16) - (20) if and only if  $\alpha_{10}, \alpha_{30} \in \mathbb{C}$  with*

$$(21) \quad |\alpha_{10}|^2 + |\alpha_{30}|^2 = 1.$$

*Proof.* ( $\Leftarrow$ ) If  $|\alpha_{10}|^2 = 1$ , then we choose  $\alpha_{21} = \alpha_{31} = 1$  and all other coefficients to be 0 to obtain a solution to Equations (16) - (20). Likewise, if  $|\alpha_{10}|^2 = 0$ , then choose  $\alpha_{11} = \alpha_{21} = 1$  and all other coefficients to be 0.

Now suppose that  $0 < |\alpha_{10}| < 1$ , and we choose  $\lambda = \frac{-\overline{\alpha_{10}}\alpha_{30}}{1 - |\alpha_{10}|^2}$ . Then choose  $\alpha_{11}$  and  $\alpha_{12}$  such that  $|\alpha_{11}|^2 + |\alpha_{12}|^2 = 1 - |\alpha_{10}|^2$ . Now let  $\alpha_{31} = \lambda\alpha_{11}$  and  $\alpha_{32} = \lambda\alpha_{12}$ . We have

$$\begin{aligned}
 (22) \quad \alpha_{10}\overline{\alpha_{30}} + \alpha_{11}\overline{\alpha_{31}} + \alpha_{12}\overline{\alpha_{32}} &= \alpha_{10}\overline{\alpha_{30}} + \overline{\lambda}|\alpha_{11}|^2 + \overline{\lambda}|\alpha_{12}|^2 \\
 &= \alpha_{10}\overline{\alpha_{30}} + \overline{\lambda}(1 - |\alpha_{10}|^2) \\
 &= 0,
 \end{aligned}$$

so Equation (19) is satisfied.

Equation (17) forces  $\alpha_{20} = 0$ ; choose  $\alpha_{21}$  and  $\alpha_{22}$  such that  $|\alpha_{21}|^2 + |\alpha_{22}|^2 = 1$  and  $\alpha_{11}\overline{\alpha_{21}} + \alpha_{12}\overline{\alpha_{22}} = 0$ . Thus, Equations (18) and (20) are satisfied. Finally, regarding Equation (16), it is satisfied for  $j = 0, 1, 2$  by construction. For  $j = 3$ , we calculate:

$$\begin{aligned}
 (23) \quad |\alpha_{30}|^2 + |\alpha_{31}|^2 + |\alpha_{32}|^2 &= |\alpha_{30}|^2 + |\lambda|^2 (|\alpha_{11}|^2 + |\alpha_{12}|^2) \\
 &= |\alpha_{30}|^2 + \frac{|\alpha_{10}|^2 |\alpha_{30}|^2}{(1 - |\alpha_{10}|^2)^2} (1 - |\alpha_{10}|^2) \\
 &= |\alpha_{30}|^2 \left( 1 + \frac{|\alpha_{10}|^2}{1 - |\alpha_{10}|^2} \right) \\
 &= \frac{|\alpha_{30}|^2}{1 - |\alpha_{10}|^2} \\
 &= 1
 \end{aligned}$$

as required.

( $\Rightarrow$ ) Suppose that we have a solution to Equations (16) - (20). If  $|\alpha_{10}| = 1$ , then we must have  $\alpha_{11} = \alpha_{12} = 0$ , and thus Equation (19) requires  $\alpha_{30} = 0$ , so Equation (21) holds.

Now suppose  $|\alpha_{10}| < 1$ . Since  $\alpha_{20} = 0$ , we must have that  $|\alpha_{21}|^2 + |\alpha_{22}|^2 = 1$ . Combining this with Equations (18) and (20) imply that the matrix

$$\begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{31} & \alpha_{32} \end{pmatrix}$$

is singular. Thus, there exists a  $\lambda$  such that  $\alpha_{31} = \lambda\alpha_{11}$  and  $\alpha_{32} = \lambda\alpha_{12}$ . Using the same computation as in Equation (22), we conclude that  $\lambda = \frac{-\overline{\alpha_{10}}\alpha_{30}}{1 - |\alpha_{10}|^2}$ ; then Equation (23) implies (21).  $\square$

The coefficient matrix we obtain from this construction is

$$H = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ \alpha_{10} + \alpha_{11} + \alpha_{12} & \alpha_{10} - \alpha_{11} + \alpha_{12} & \alpha_{10} - \alpha_{11} - \alpha_{12} & \alpha_{10} + \alpha_{11} - \alpha_{12} \\ \alpha_{21} + \alpha_{22} & -\alpha_{21} + \alpha_{22} & -\alpha_{21} - \alpha_{22} & \alpha_{21} - \alpha_{22} \\ \alpha_{30} + \lambda\alpha_{11} + \lambda\alpha_{12} & \alpha_{30} - \lambda\alpha_{11} + \lambda\alpha_{12} & \alpha_{30} - \lambda\alpha_{11} - \lambda\alpha_{12} & \alpha_{30} + \lambda\alpha_{11} - \lambda\alpha_{12} \end{pmatrix}$$

where we are allowed to choose  $\alpha_{11}$ ,  $\alpha_{12}$ ,  $\alpha_{21}$  and  $\alpha_{22}$  subject to the normalization condition in Equation (16). However, those choices do not affect the construction, since if we apply Proposition 1 and the calculation from Theorem 2, we obtain

$$(24) \quad P_V S_\omega \mathbf{1} = (\alpha_{10})^{\ell_1(n)} \cdot (0)^{\ell_2(n)} \cdot (\alpha_{30})^{\ell_3(n)} e^{2\pi i n x}.$$

This will in fact be a Parseval frame for  $L^2(\mu_4)$ , provided  $V \subset \mathcal{K}$ , as in the proof of Theorem 2.

**Theorem 3.** *Suppose  $p, q \in \mathbb{C}$  with  $|p|^2 + |q|^2 = 1$ . Then*

$$\{p^{\ell_1(n)} \cdot 0^{\ell_2(n)} \cdot q^{\ell_3(n)} e^{2\pi i n x} : n \in \mathbb{N}_0\}$$

*is a Parseval frame for  $L^2(\mu_4)$ , provided  $p \neq 0$ .*

*Proof.* Substitute  $\alpha_{10} = p$  and  $\alpha_{30} = q$  in Proposition 2 and Equation (24). As noted, we only need to verify  $V \subset \mathcal{K}$ . We proceed as in the proof of Theorem 2; indeed, define  $f$ ,  $h_X$ ,  $\mathbf{m}_j$  and  $g_j$  as previously. We obtain  $b_0 = 1$ ,  $b_1 = \overline{p}$ ,  $b_2 = 0$ , and  $b_3 = \overline{q}$ , so Equation (8) becomes

$$h_X(t) = \cos^2\left(\frac{\pi t}{2}\right) h_X\left(\frac{t}{4}\right) + |\overline{p}|^2 \sin^2\left(\frac{\pi t}{2}\right) h_X\left(\frac{t-1}{4}\right) + |\overline{q}|^2 \sin^2\left(\frac{\pi t}{2}\right) h_X\left(\frac{t-3}{4}\right).$$

From here, the same argument shows that  $h_X \equiv 1$ , and  $V \subset \mathcal{K}$ .  $\square$

## 5. CONCLUDING REMARKS

We remark here that the constructions given above for  $\mu_4$  does not work for  $\mu_3$ . Indeed, we have the following no-go result. To obtain the measure  $\mu_3 \times \lambda$ , we consider the iterated function system:

$$\Upsilon_0(x, y) = \left(\frac{x}{3}, \frac{y}{2}\right), \quad \Upsilon_1(x, y) = \left(\frac{x+2}{3}, \frac{y}{2}\right), \quad \Upsilon_2(x, y) = \left(\frac{x}{3}, \frac{y+1}{2}\right), \quad \Upsilon_3(x, y) = \left(\frac{x+2}{3}, \frac{y+1}{2}\right).$$

Using the same choice of filters, the matrix  $\mathcal{M}(x, y)$  reduces to

$$H = \begin{pmatrix} a_{00} & a_{01} & a_{02} & a_{03} \\ a_{10} & e^{4\pi i/3} a_{11} & a_{12} & e^{4\pi i/3} a_{13} \\ a_{20} & e^{2\pi i/3} a_{21} & a_{22} & e^{2\pi i/3} a_{23} \\ a_{30} & a_{31} & a_{32} & a_{33} \end{pmatrix}$$

which we require to be unitary. Additionally, we require the same conditions as for  $\mu_4$ , namely, the first row of  $H$  must have all entries  $\frac{1}{2}$ , and  $a_{j0} + a_{j2} = a_{j1} + a_{j3}$ . The inner product of the first two rows must be 0. Hence,

$$\frac{1}{2} (a_{10} + e^{4\pi i/3} a_{11} + a_{12} + e^{4\pi i/3} a_{13}) = \frac{1}{2} (a_{10} + a_{12}) (1 + e^{4\pi i/3}) = 0.$$

Consequently,  $a_{10} + a_{12} = 0$ . Likewise,  $a_{20} + a_{22} = a_{30} + a_{32} = 0$ . As a result,

$$H \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a_{00} + a_{02} \\ a_{10} + a_{12} \\ a_{20} + a_{22} \\ a_{30} + a_{32} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

and so  $H$  cannot be unitary.

It may be possible to extend the construction for  $\mu_4$  to  $\mu_3$  by considering a representation of  $\mathcal{O}_n$  for some sufficiently large  $n$ , or by considering  $\mu_3 \times \rho$  for some other fractal measure  $\rho$  rather than  $\lambda$ .

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